FULL-WAVE ANALYSIS OF A FINITE PIECE OF METAMATERIAL

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Abstract

This contribution concerns the homogenization of a finite piece of metamaterial consisting of nonoverlapping spheres. The proposed technique is based upon a full-wave analysis of a spherical sample which is accelerated with the Stable Plane Wave Method. Hereby, the scattered field of the spherical sample is determined for various incoming fields, so that the T-matrix of the sample can be determined. This T-matrix is then compared to the known analytical expression for the T-matrix of a homogeneous sphere, yielding values for the permeability and permittivity. Also, it is shown that the special properties of the analytical solution can be used to obtain these parameters directly.

1 Homogenization of a collection of spheres

In the study of metamaterials one of the central questions is which effective parameters correspond to a given microstructure. Homogenization formulas such as the Maxwell-Garnett [1] formula are only valid for low particle density and simple types of particles. In order to be able to homogenize more complicated metamaterials, for example when the spherical particles contain wire structures, a more general approach is needed. In this contribution, a very general method is proposed to homogenize metamaterials which consist of nonoverlapping spherical inclusions which are embedded in a uniform background medium. The only requirement is that all the particles can be circumscribed with spheres, such that none of the spheres overlap.

As the first stage of the method, a full-wave analysis of a spherical sample of the metamaterial (an example is shown in Figure 2) under scrutiny is performed for incoming fields of the following forms:

$$\mathbf{E}_{lm}^{inc,1}(\mathbf{r}) = \frac{\hat{\mathbf{L}}\left[j_l(kr)Y_{lm}(\mathbf{r})\right]}{\sqrt{l(l+1)}} \qquad \mathbf{E}_{lm}^{inc,2}(\mathbf{r}) = \frac{1}{k}\nabla \times \mathbf{E}_{lm}^{inc,1}(\mathbf{r}) \tag{1}$$

where $\hat{\mathbf{L}}$ is the angular momentum operator:

$$\hat{\mathbf{L}} = -j\mathbf{r} \times \nabla = j\left[\mathbf{e}_\theta \frac{1}{\sin\theta}\frac{d}{d\theta} - \mathbf{e}_\phi \frac{d}{d\phi}\right] \tag{2}$$

The resulting scattered fields can be decomposed into functions similar to (1), but with spherical Hankel functions instead of spherical Bessel functions. The coefficients arising in this decomposition can be interpreted as entries of the T-matrix of the *entire sample* of metamaterial. If this analysis is performed for sufficiently large l, then any incoming field (of which the sources are not located inside the piece of metamaterial) can be reconstructed as a linear combination of the above $\mathbf{E}_{lm}^{inc,1}(\mathbf{r})$ and $\mathbf{E}_{lm}^{inc,2}(\mathbf{r})$. Since the solution of the scattering problem is known for all these fields, the solution is immediately known for any incoming field. This fact demonstrates that the T-matrix contains all the information necessary to determine the behavior of the piece of metamaterial, and thus also contains the effective material parameters.

The second stage of the method is the matching of the T-matrix to its analytical expression. The T-matrix of a homogeneous sphere is diagonal and the diagonal elements are given by:

$$T^1_{l,m} = -\frac{Z_i \frac{\mathcal{J}_l(k_o a)}{j_l(k_o a)} - Z_o \frac{\mathcal{J}_l(k_i a)}{j_l(k_i a)}}{Z_i \frac{\mathcal{H}^{(2)}_l(k_o a)}{j_l(k_o a)} - Z_o \frac{\mathcal{J}_l(k_i a)}{j_l(k_i a)} \frac{h^{(2)}_l(k_o a)}{j_l(k_o a)}} \tag{3}$$

$$T^2_{l,m} = -\frac{Z_o \frac{\mathcal{J}_l(k_o a)}{j_l(k_o a)} - Z_i \frac{\mathcal{J}_l(k_i a)}{j_l(k_i a)}}{Z_o \frac{\mathcal{H}^{(2)}_l(k_o a)}{j_l(k_o a)} - Z_i \frac{\mathcal{J}_l(k_i a)}{j_l(k_i a)} \frac{h^{(2)}_l(k_o a)}{j_l(k_o a)}} \tag{4}$$

Here, $\mathcal{J}_l(x) = \frac{1}{x}\frac{d}{dx}\left[x j_l(x)\right]$ and $\mathcal{H}^{(2)}_l(x) = \frac{1}{x}\frac{d}{dx}\left[x h^{(2)}_l(x)\right]$. The unknowns are Z_i and k_i, the impedance and wavenumber inside the sphere. The left hand sides of both of these equations are known as are the radius of the piece of metamaterial and the parameters of the surrounding background medium. Therefore these equations can be solved for the two quantities $\frac{Z_o}{Z_i}$ (thus yielding Z_i) and $A_l = \frac{\mathcal{J}_l(k_i a)}{j_l(k_i a)}$. From the latter, a unique value for $k_i a$ is not easily found, but since this quantity is known for a whole series of l, the recurrences of the Bessel functions can be used to obtain the following quadratic equation which can be solved easily:

$$-\left(\frac{l+1}{k_i a}\right)^2 + (A_l - A_{l+1})\frac{l+1}{k_i a} + A_l A_{l+1} + 1 = 0 \tag{5}$$

Determining which one of the two roots to choose is done by calculating these roots for various l and checking which one is consistent.

2 Full-wave analysis with the Stable Plane Wave Method

The largest computational burden of the method is in the full-wave analysis of the piece of metamaterial. Indeed, for each of the incoming fields (1), a scattering problem must be solved. If each particle is discretized with all of its geometrical detail, the amount of unknowns becomes too large to handle. Therefore the requirement that all the particles can be circumscribed with spheres, such that none of the spheres overlap, was imposed. This allows each of the particles to be described by means of a T-matrix, which greatly reduces the number of unknowns N. However, this reduction still does not make the simulation of a realistic sample of metamaterial possible. Therefore, an iterative scheme was used, together with the Stable Plane Wave Method to accelerate the associated matrix-vector products. The Stable Plane Wave Method is a so-called Fast Multipole Method (FMM), which breaks up the computational domain into groups. The fields generated by sources in one group (the sending group) are aggregated into a radiation pattern for this group, then translated and disaggregated onto the receiving group (See Figure 1). Thus in such a scheme, the groups effectively interact as a whole. This is a lot less computationally intensive than a direct matrix-vector multiplication, it even reduces the complexity of a matrix-vector multiplication from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$.

The specific choice for the Stable Plane Wave Method is motivated by the fact that the geometrical detail of metamaterials is situated at a scale which is significantly smaller than one wavelength, while a piece of metamaterial as a whole can very well be larger than one wavelength. Because of this, an FMM is needed which is numerically stable for all frequencies. The Stable Plane Wave Method satisfies this requirement by using both evanescent and propagating plane waves, but it has a drawback. Its representation of the Green function only converges in one half-space of choice. This means that six representations are needed to cover the entire space and as a consequence, six aggregations and disaggregations are needed. This problem has been recently solved [2] by first rotating to a frame where the former $(1, 1, 1)$-axis is the new z axis. This results in a reduction of the computational cost by a factor of 6.

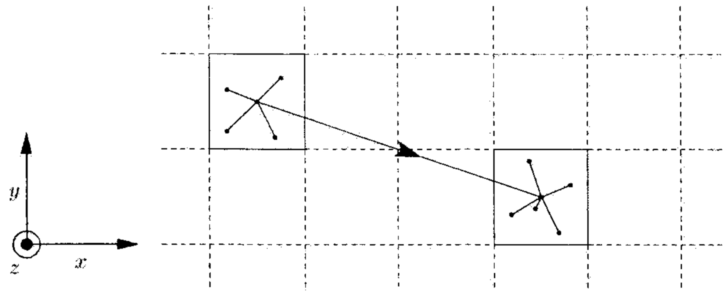

Figure 1: Two interacting groups in a FMM.

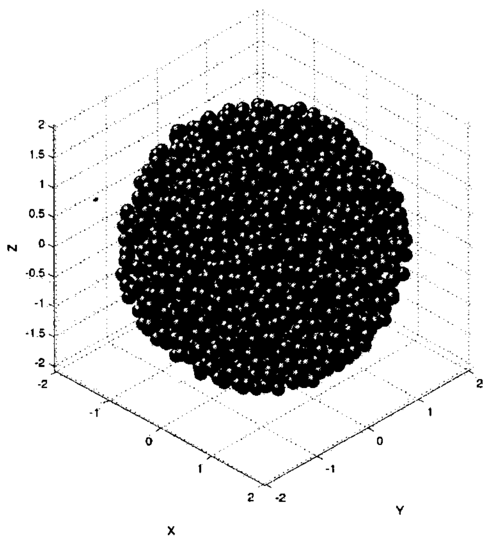

Figure 2: A spherical sample with 2000 spheres.

3 An example

In Figure 2, a spherical sample of metamaterial is shown which has been homogenized with the method described above. It consists of 2000 homogeneous spheres with radius $0.1m$ and parameters $\varepsilon = 2 - 0.2j$ and $\mu = 1.5 - 0.1j$. The frequency is $5 \times 10^7 Hz$ and the radius of the sample sphere is $2m$. The results were $\varepsilon^{eff} = 1.19 - j0.03$ and $\mu^{eff} = 1.11 - j0.019$. To validate our technique, the homogenization was also done with the Maxwell-Garnett formula. This yields the following results: $\varepsilon^{eff,MG} = 1.201 - j0.03195$ and $\mu^{eff,MG} = 1.112 - j0.01974$. However, note that while this particular homogenization could have been done with the Maxwell-Garnett formula, our method is far more general.

References

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